It is also possible to define the arithmetic-geometric mean for complex numbers. To do this, we first must make the geometric mean unambiguous by choosing a branch of the square root. We may do this as follows: Let $a$ and $b$ be two non-zero complex numbers such that $a \neq sb$ for any real number $s < 0$. Then we will say that $c$ is the geometric mean of $a$ and $b$ if $c^2 = ab$ and $c$ is a convex combination of $a$ and $b$ (i.e. $c = sa + tb$ for positive real numbers $s$ and $t$).

Geometrically, this may be understood as follows: The condition $a \neq sb$ means that the angle between $0a$ and $0b$ differs from $\pi$. The square root of $ab$ will lie on a line bisecting this angle, at a distance $\sqrt{|ab|}$ from 0. Our condition states that we should choose $c$ such that $0c$ bisects the angle smaller than $\pi$, as in the figure below:

Analytically, if we pick a polar representation $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$ with $|\alpha - \beta| < \pi$, then $c = \sqrt{|ab|}e^{i\frac{\alpha + \beta}{2}}$. Having clarified this preliminary item, we now proceed to the main definition.

As in the real case, we will define sequences of geometric and arithmetic means recursively and show that they converge to the same limit. With our convention, these are defined as follows:

$$g_0 = a$$
$$a_0 = b$$
$$g_{n+1} = \sqrt{a_ng_n}$$
$$a_{n+1} = \frac{a_n + g_n}{2}$$

We shall first show that the phases of these sequences converge. As above, let us define $\alpha$ and $\beta$ by the conditions $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$, and $|\alpha - \beta| < \pi$. 

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*Complex Arithmetic Geometric Mean* created: ⟨2013-03-21⟩ by ⟨rspuzio⟩ version: ⟨39480⟩ Privacy setting: ⟨1⟩ (Result) ⟨33E05⟩ ⟨36E60⟩

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Suppose that $z$ and $w$ are any two complex numbers such that $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$ with $|\phi - \theta| < \pi$. Then we have the following:

- The phase of the geometric mean of $z$ and $w$ can be chosen to lie between $\theta$ and $\phi$. This is because, as described earlier, this phase can be chosen as $(\theta + \phi)/2$.

- The phase of the arithmetic mean of $z$ and $w$ can be chosen to lie between $\theta$ and $\phi$.

By a simple induction argument, these two facts imply that we can introduce polar representations $a_n = |a_n|e^{i\theta_n}$ and $g_n = |g_n|e^{i\phi_n}$ where, for every $n$, we find that $\theta_n$ lies between $\alpha$ and $\beta$ and likewise $\phi_n$ lies between $\alpha$ and $\beta$. Furthermore, since $\phi_{n+1} = (\phi_n + \theta_n)/2$ and $\theta_{n+1}$ lies between $\phi_n$ and $\theta_n$, it follows that

$$|\phi_{n+1} - \theta_{n+1}| \leq \frac{1}{2}|\phi_n - \theta_n|.$$ 

Hence, we conclude that $|\phi_n - \theta_n| \to 0$ as $n \to \infty$. By the principle of nested intervals, we further conclude that the sequences $\{\theta_n\}_{n=0}^{\infty}$ and $\{\phi_n\}_{n=0}^{\infty}$ are both convergent and converge to the same limit.

Having shown that the phases converge, we now turn our attention to the moduli. Define $m_n = \max(|a_n|, |g_n|)$. Given any two complex numbers $z, w$, we have

$$|\sqrt{zw}| \leq \max(|z|, |w|)$$

and

$$\left|\frac{z + w}{2}\right| \leq \max(|z|, |w|),$$

so this sequence $\{m_n\}_{n=0}^{\infty}$ is decreasing. Since it bounded from below by 0, it converges.

Finally, we consider the ratios of the moduli of the arithmetic and geometric means. Define $x_n = |a_n|/|g_n|$. As in the real case, we shall derive a recursion relation for this quantity:

$$x_{n+1} = \frac{|a_{n+1}|}{|g_{n+1}|} = \frac{|a_n + g_n|}{2\sqrt{|a_ng_n|}} = \frac{\sqrt{|a_n|^2 + 2|a_n||g_n| \cos(\theta_n - \phi_n) + |g_n|^2}}{2\sqrt{|a_ng_n|}} = \frac{1}{2} \sqrt{\frac{|a_n|}{|g_n|} + 2 \cos(\theta_n - \phi_n) + \frac{|g_n|}{|a_n|}} = \frac{1}{2} \sqrt{x_n + 2 \cos(\theta_n - \phi_n) + \frac{1}{x_n}}.$$
For any real number $x \geq 1$, we have the following:

\[
\begin{align*}
    x - 1 & \geq 0 \\
    (x - 1)^2 & \geq 0 \\
    x^2 - 2x + 1 & \geq 0 \\
    x^2 + 1 & \geq 2x \\
    x + \frac{1}{x} & \geq 2
\end{align*}
\]

If $0 < x < 1$, then $1/x > 1$, so we can switch the roles of $x$ and $1/x$ and conclude that, for all real $x > 0$, we have

\[
x + \frac{1}{x} \geq 2.
\]

Applying this to the recursion we just derived and making use of the half-angle identity for the cosine, we see that

\[
x_{n+1} \geq \frac{1}{2} \sqrt{2 + 2 \cos(\theta_n - \phi_n)} = \cos \left( \frac{\theta_n - \phi_n}{2} \right).
\]